A CLASS OF PLANAR FOUR-COLORABLE GRAPHS

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ABSTRACT

A sufficient condition is given for a planar graph to be 4-colorable. This condition is in terms of the sums of the degrees of a subset of the vertex set of the graph.

1. Let G be a planar graph on n vertices, without loops or multiple edges. G is said to be k-colorable if the vertex set $V(G)$ of G can be partitioned into k mutually disjoint subsets $S_1,S_2,\cdots,S_k$ such that no two adjacent vertices lie in the same subset S_i. In this paper, V_r denotes a set of r vertices from $V(G)$ and q_r is the sum of the degrees of the vertices of V_r. Our purpose is to establish the following sufficient condition for a planar graph to be 4-colorable.

THEOREM 1. *Let G be a planar graph on n vertices.* If *V_r is a subset of $V(G)$ such that $r \leqq 14$ and $q_r > 2n - 12 + 4r$, then G is 4-colorable.*

In [1], Malec and Skupién establish the following simple consequence of Euler's formula for planar graphs:

THEOREM 2. *Let d be an integer such that $2 \leqq d \leqq 5$. If G is planar on n-vertices, V_r a subset of $V(G)$ with $q_r > 6n - 12 - d(n - r)$, then G has a vertex of degree $d - 1$ or less.*

As a consequence of this theorem, they prove

THEOREM 3. *If G is planar on n vertices and if G has a vertex of degree $n - 6$ or larger, then G is 4-colorable.*

2. Although Malec and Skupién do not state this explicitly in Theorem 2, it is clear that when $r < n$, the vertex of degree $d - 1$ or less may be taken not to be in V_r. As we have $r \leqq 14$ in Theorem 1, we may as well assume $r < n$, for otherwise G has 14 or fewer vertices and is therefore 4-colorable.

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PROOF OF THEOREM 1. Let $V_r \subseteq V(G)$ be the set $V_r = \{x_1,\cdots,x_r\}$. We will define a sequence $\{G_i\}$ of maximal planar graphs where G_{i+1} is obtained from G_i by deleting a vertex y_i of degree 3 or 4, where $y_i \notin V_r$. If $v \in V(G_i)$, let $\rho_i(v)$ denote the degree of v in G_i and define $Q_i = \rho_i(x_1) + \cdots + \rho_i(x_r)$. As a first step, add sufficient edges to G to form a graph G_0 which is maximal planar and note $Q_0 (= \sum_{i=1}^{r} \rho_{G_0}(x_i)) \geqq q_r$.

We now describe the way to form G_1; the same procedure is used as an algorithm to form $G_2, G_3 \cdots$. The hypotheses of Theorem 1, together with Theorem 2, guarantee the existence of a vertex of degree 3 in G_0, where this vertex is not in V_r. This insures that the algorithm can be started.

We will choose a vertex y_0 in G_0 according to the following priorities:

(1) $\rho_0(y_0) = 4$, $y_0 \notin V_r$

(2) $\rho_0(y_0) = 3$, $y_0 \notin V_0$, y_0 adjacent to no more than 2 vertices of V_r

(3) $\rho_0(y_0) = 3$, $y_0 \notin V_r$, y_0 adjacent to 3 vertices of V_r.

Form G_1 as follows:

Case 1. Let $\rho_0(y_0) = 4$, $y_0 \notin V_r$. If y_0 is adjacent to 2 or fewer vertices of V_r, delete y_0 and its incident edges to form a graph H_1. Add a sufficient set of edges to H_1 to form G_1, where G_1 is maximal planar. Note G_1 has $n-1$ vertices and $Q_1 \geqq Q_0 - 2 > 2(n-1) - 12 + 4r$.

If y_0 is adjacent to 3 vertices of V_r, say a, b and c, we have in G_0 the following Figure:

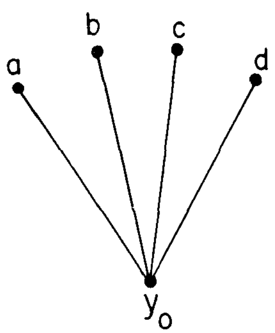

Fig. 1

If there is an edge from a to c in G_0, then there can be no edge from b to d in G_0. Form H_1 by deleting y_0 and its incident edges from G_0. Add the edge (b,d) to H_1 (this is clearly possible) together with sufficient other edges to form a maximal planar graph G_1. Again, we note $Q_1 \geqq Q_0 - 2$. If the edge (a,c) is not in G_0, we add

(a,c) to H_1 and then form a maximal planar graph G_1 where, as above, $Q_1 \geqq Q_0 - 2$.

If y_0 is adjacent to 4 vertices of V_r, say a, b, c and d, then we may add either the edge (a,c) or the edge (b,d) to H_1. We then add sufficiently many more edges to obtain a maximal planar graph G_1, where $Q_1 \geqq Q_0 - 2 > 2(n-1) - 12 + 4r$.

Case 2. Suppose G_0 has no vertices of degree 4 not in V_r. Search for a vertex y_0 of degree 3, such that y_0 is not in V_r and y_0 has no more then 2 neighbors in V_r. Removing y_0 and its incident edges will give a maximal planar graph G_1 with $n-1$ vertices such that $Q_1 > 2(n-1) - 12 + 4r$.

Case 3. Suppose now that G_0 has no vertices of degree 4 in V_r and suppose all vertices of degree 3 not in V_r are joined to 3 vertices of V_r. Recall, G_0 has at least one vertex y_0 of degree 3 not in V_r.

If $Q_0 = 2n - 12 + 4r + 1$, the algorithm terminates. Otherwise, remove the vertex y_0 and its incident edges to form the maximal planar graph G_1 and note $Q_1 = Q_0 - 3 > 2(n-1) - 12 + 4r$. We form G_2 from G_1 just as we formed G_1 from G_0, searching first vertices of type (1), then of type (2) and finally of type (3). At each stage of the algorithm, we have a maximal planar graph G_j with $(n-j)$ vertices and where $Q_j > 2(n-j) - 12 + 4r$.

Continue until the algorithm terminates with the graph G_i. In G_i we have:

(a) there are no vertices of degree 4 in G_i that are in V_r

(b) any vertex of degree 3 not in V_r has all its neighbors in V_r

(c) $Q_i = 2(n-i) - 12 + 4r + 1$

(d) G_i has $n - i$ vertices.

Using Euler's formula, in which j is the number of vertices of degree 3 not in V_r, we obtain

$$6(n-i) - 12 \geqq 3j + 2(n-i) - 12 + 4r + 1 + 5(n-i-j-r)$$

$$= 7(n-i) - 2j - r - 11$$

so that $j + r \geqq n - i + 1 - j$.

We now establish that $j \leqq 2(r-2)$. Form the section graphs $G_i(V_r)$, i.e. the graph consisting of the r vertices of V_r together with the edges of G_i each of whose endpoints is in V_r. By Euler's formula, $G_i(V_r)$ has $2(r-2)$ or fewer faces. As G_i is maximal planar, the neighbors of a vertex of degree 3 form a circuit of length 3 in G_i. Therefore, if $\rho_i(y) = 3$ and if $y \notin V_r$, then we have from (b) that y lies in a triangular face of $G_i(V_r)$ when we regard $G_i(V_r)$ as a subgraph of G_i. Furthermore, as G_i is maximal planar, no other vertex of G_i can lie in the same triangular face of $G_i(V_r)$ as does y, or we would have $\rho_i(y) \geqq 4$. This establishes the inequality $j \leqq 2(r-2)$, for any vertex of degree 3 not in V_r lies in a triangular face of $G_i(V_{ir})$ and no triangular face has more than one such vertex.

From before, $j + r \geqq n - i + 1 - j$, so we obtain $3r - 5 \geqq (n-i) - j$. Since $r \leqq 14$, the inequality $37 \geqq (n-i)-j$ follows. Form G_{i+j} from G_i by deleting the j vertices of degree 3 and their incident edges. Since G_{i+j} has 37 or fewer vertices, it can be 4-colored [4]. Since G_{k+1} was formed by deleting a vertex of degree 4 or less from G_k, it is clear that when G_{k+1} is 4-colorable then so is G_k (cf. the dual statement of theorem 6.4.4 of [3]). Thus, G_0 is 4-colorable and so therefore is G.

We should, perhaps, mention the obvious; that is, that Theorem 1 places upper bounds on the possible degrees of the vertices in a 5-chromatic planar graph. For example, if G is planar with n vertices and not 4-colorable, then G cannot have 3 vertices x_1, x_2 and x_3 such that $\rho(x_1) + \rho(x_2) + \rho(x_3) > 2n$. Thus, if there is a planar graph G with 40 vertices which is 5-chromatic, G cannot have 3 vertices having degree 27 or more. As another example, suppose G has 49 vertices with $V(G) = \{x_1, x_2, \cdots, x_{49}\}$ where $\rho(x_i) \geqq \rho(x_i) \geqq \rho(x_{i+1})$. If G is planar and not 4-colorable, then $\rho(x_i) \leqq 11$, $i = 14, \cdots, 49$.

References

1. M. Malec and Z. Skupién, *On the maximal planar graphs and the four colour problem*, Prace Mat. **12** (1969), 205–209.

2. O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ. 38, Providence, 1962.

3. O. Ore, *The Four-Color Problem*, Academic Press, New York, 1967.

4. O. Ore and J. Stemple, *Numerical calculations on the four-color problem*, J. Combinatorial Theory, **8** (1970), 65–78.

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